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Simpson-like and Hermite-Hadamard-like Type Inequalities for Harmonically Quasi-convex Functions

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Abstract

In this paper, by using a generalized integral identity for differentiable functions, the author obtain some new upper bounds of Hermite-Hadamard type inequalities and new Simpson-like type inequalities, for differentiable harmonically quasi-convex functions.

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1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Hermite-Hadamard's inequalities for convex functions have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1, 2, 3, 7, 8, 17] and references therein.

Let us recall some definitions of several kinds of convex functions:

Definition 1. Let I be an interval in R. Then $f : I \to R$ is said to be convex on I if the inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{1}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf\left(y\right) + (1-t)\,f\left(x\right) \tag{2}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

In [4], Imdat Işcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

Theorem 1.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically convex function on an interval I and $f \in L[a, b]$, where $a, b \in I$ with $a < b$.*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Also, in [4, 5], Imdat Işcan established some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

Theorem 1.2. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\mid f' \mid^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2 \mid f'(a) \mid^q + \lambda_3 \mid f'(b) \mid^q\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),\\
\lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),\\
\lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)\\
&= \lambda_1 - \lambda_2.
\end{aligned}$$

In this article we consider the following special functions:

Definition 3. The hypergeometric function ${}_2F_1[a,b,c,x]$ is defined for $\mid x \mid< 1$ by the power series

$$ {}_2F_1[a,b,c,x] = \Sigma_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}. \tag{4}$$

Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$(q)_n = \begin{cases} 1, & n=0 \\ q(q+1)\cdots(q+n-1), & n>0. \end{cases}$$

Definition 4. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t. \tag{5}$$

In this paper, we give some generalized inequalities connected with the left and right parts of (3), as a result of this, we obtain some generalized Simpson-like and Hermite-Hadamard-like type inequalities for differentiable harmonically quasi-convex functions by using an integral identity for differentiable functions.

2 Main results

In order to find some new inequalities of Hermite-Hadamard-like and Simpson-like type inequalities connected with the left and right parts of (3) for functions whose derivatives are harmonically quasi-convex, we need the following lemma [15]:

Lemma 1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. Then, for $h \in (0, 1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ for any $n \geq 2$ the following identity*

$$\begin{aligned}
&S_a^b(f)(h, n) \\
&\equiv \frac{1}{n}\Big[f(a) + (n-2)f\Big(\frac{ab}{A_h(a, b)}\Big) + f(b)\Big] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
&= ab(a-b)\int_0^1 p(t, h)\frac{1}{A_t^2(a, b)}f'\Big(\frac{ab}{A_t(a, b)}\Big)\mathrm{d}t
\end{aligned} \tag{6}$$

holds, for $t \in [0, 1]$, where $A_t(a, b) = (1-t)a + tb$ and

$$p(t, h) = \begin{cases} t - \frac{1}{n} & t \in [0, h] \\ t - \frac{n-1}{n} & t \in (h, 1]. \end{cases}$$

Proof By the simple calculation, this is proved.

Now we turn our attention to establish inequalities of Hermit-Hadamard-like and Simpson-like type for differentiable harmonically convex functions.

Theorem 2.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is harmonically quasi-convex on $[a, b]$, then, for $h \in (0, 1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ for any $n \geq 2$ the following inequality*

$$\begin{aligned}
&\big|S_a^b(f)(h, n)\big| \\
&\leq ab(b-a)\big\{\mu_{11}(h, n) + \mu_{12}(h, n)\big\}\sup\big\{\big|f'(a)\big|, \big|f'(b)\big|\big\}
\end{aligned} \tag{7}$$

holds, where

$$\begin{aligned}
\mu_{11}(h, n) &= \frac{(2-h-nh)a + hb}{na(b-a)A_h(a, b)} + \frac{1}{(b-a)^2}\ln\Big[\frac{aA_h(a, b)}{A_{\frac{1}{n}}^2(a, b)}\Big], \\
\mu_{12}(h, n) &= -\frac{(1-h)a + (1+h-n+nh)b}{nb(b-a)A_h(a, b)} \\
&\qquad + \frac{1}{(b-a)^2}\ln\Big[\frac{bA_h(a, b)}{A_{\frac{1}{n}}^2(a, b)}\Big].
\end{aligned}$$

Proof From Lemma 1, since $|f'|$ is harmonically convex on $[a, b]$, we have

$$\begin{aligned}
&\big|S_a^b(f)(h, n)\big| \\
&\leq ab(b-a)\int_0^h \frac{\big|t - \frac{1}{n}\big|}{A_t^2(a, b)}\Big|f'\Big(\frac{ab}{A_t(a, b)}\Big|\Big)\mathrm{d}t \\
&\qquad + \int_h^1 \frac{\big|t - \frac{n-1}{n}\big|}{A_t^2(a, b)}\Big|f'\Big(\frac{ab}{A_t(a, b)}\Big)\Big|\mathrm{d}t \\
&\leq \Big\{\int_0^h \frac{\big|t - \frac{1}{n}\big|}{A_t^2(a, b)}\mathrm{d}t + \int_h^1 \frac{\big|t - \frac{n-1}{n}\big|}{A_t^2(a, b)}\mathrm{d}t\Big\}\sup\Big\{\Big|f'(a)\Big|, \Big|f'(b)\Big|\Big\}.
\end{aligned} \tag{8}$$

Note that

$$
\begin{aligned}
&(a) \int_0^h \frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\mathrm{d}t \\
&= \frac{(2-h-nh)a+hb}{na(b-a)A_h(a,b)} + \frac{1}{(b-a)^2}\ln\Big[\frac{aA_h(a,b)}{A_{\frac{1}{n}}^2(a,b)}\Big], \qquad (9)\\
&(b) \int_h^1 \frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\mathrm{d}t \\
&= -\frac{(1-h)a+(1+h-n+nh)b}{nb(b-a)A_h(a,b)} \\
&\qquad + \frac{1}{(b-a)^2}\ln\Big[\frac{bA_h(a,b)}{A_{\frac{1}{n}}^2(a,b)}\Big]. \qquad (10)
\end{aligned}
$$

By substituting (9) and (10) in (8), we get the desired result.

Corollary 2.1. *In Theorem 2.1, one has:*

$$
\begin{aligned}
&\left|S_a^b(f)(\frac{1}{2},6)\right| \\
&\leq ab(b-a)\big\{\mu_{11}(\frac{1}{2},6)+\mu_{12}(\frac{1}{2},6)\big\}\sup\big\{\left|f'(a)\right|,\left|f'(b)\right|\big\}.
\end{aligned}
$$

Theorem 2.2. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for $h \in (0,1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ for any $n \geq 2$ the following inequality*

$$
\begin{aligned}
&\left|S_a^b(f)(h,n)\right| \\
&\leq ab(b-a)\Big\{\big(\mu_{21}(h,n)+\mu_{22}(h,n)\big)^{\frac{1}{p}}h^{\frac{1}{q}} \\
&\qquad + \big(\mu_{23}(h,n)+\mu_{24}(h,n)\big)^{\frac{1}{p}}(1-h)^{\frac{1}{q}}\Big\} \\
&\qquad \times\Big(\sup\big\{\ |\ f'(a)\ |^q, |\ f'(b)\ |^q\ \big\}\Big)^{\frac{1}{q}}. \qquad (11)
\end{aligned}
$$

holds, where

$$\begin{aligned}
\mu_{21}(h,n) &= \frac{1}{(1+p)n^{1+p}a^{2p}}\,{}_2F_1[1,2p,2+p,-\frac{b-a}{na}],\\
\mu_{22}(h,n) &= A_{\frac{1}{n}}^{1-2p}(a,b)\frac{n^p}{(a-b)^{1+p}}\beta(1-2p,1+p)\\
&\quad+\frac{A_h^{1-2p}(a,b)A_{\frac{1}{n}}^{p}(a,b)}{(2p-1)(a-b)^{1+p}}\,{}_2F_1[1-2p,-p,2-2p,\frac{A_h(a,b)}{A_{\frac{1}{n}}(a,b)}],\\
\mu_{23}(h,n) &= A_{\frac{1}{n}}^{1-p}(b,a)\frac{\beta(1-2p,1+p)}{(b-a)^{1+p}}\\
&\quad+\frac{A_h^{1-2p}(a,b)}{2p-1}\frac{A_{\frac{1}{n}}^{p}(b,a)}{(b-a)^{1+p}}\,{}_2F_1[1-2p,-p,2-2p,\frac{A_h(a,b)}{A_{\frac{1}{n}}(b,a)}],\\
\mu_{24}(h,n) &= A_{\frac{1}{n}}^{1-p}(b,a)\frac{\beta(1-2p,1+p)}{(a-b)^{1+p}}\\
&\quad+\frac{b^{1-2p}A_{\frac{1}{n}}^{p}(b,a)}{(2p-1)(a-b)^{1+p}}\,{}_2F_1[1-2p,-p,2-2p,\frac{b}{A_{\frac{1}{n}}(b,a)}].
\end{aligned}$$

Proof From Lemma 1 and by the Hölder integral inequality, we have

$$\begin{aligned}
&\left|S_a^b(f)(h,n)\right|\\
&\le ab(b-a)\Big[\int_0^h \frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t\\
&\quad+\int_h^1 \frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t\Big]\\
&\le ab(b-a)\Big[\Big\{\int_0^h \frac{\left|t-\frac{1}{n}\right|^p}{A_t^{2p}(a,b)}\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^h \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\\
&\quad+\Big\{\int_h^1 \frac{\left|t-\frac{n-1}{n}\right|^p}{A_t^{2p}(a,b)}\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_h^1 \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (12)
\end{aligned}$$

Note that

$$\begin{aligned}
(a)&\int_0^h \frac{\left|t-\frac{1}{n}\right|^p}{A_t^{2p}(a,b)}\mathrm{d}t\\
&=\int_0^{\frac{1}{n}} \frac{(\frac{1}{n}-t)^p}{A_t^{2p}(a,b)}\mathrm{d}t+\int_{\frac{1}{n}}^h \frac{(t-\frac{1}{n})^p}{A_t^{2p}(a,b)}\mathrm{d}t\\
&=\mu_{21}(h,n)+\mu_{22}(h,n), \qquad (13)
\end{aligned}$$

$$
\begin{aligned}
&(b) \int_h^1 \frac{\left|t-\frac{n-1}{n}\right|^p}{A_t^{2p}(a,b)}\mathrm{d}t \\
&= \int_h^{\frac{n-1}{n}} \frac{(\frac{n-1}{n}-t)^p}{A_t^{2p}(a,b)}\mathrm{d}t + \int_{\frac{n-1}{n}}^1 \frac{(t-\frac{n-1}{n})^p}{A_t^{2p}(a,b)}\mathrm{d}t \\
&= \mu_{23}(h,n) + \mu_{24}(h,n). \qquad (14)
\end{aligned}
$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$, we have

$$
\begin{aligned}
&(i) \int_0^h \left|f'\Big(\frac{ab}{A_t(a,b)}\Big)\right|^q \mathrm{d}t \\
&\leq \int_0^h \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\mathrm{d}t \\
&= h \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}, \qquad (15) \\
&(ii) \int_h^1 \left|f'\Big(\frac{ab}{A_t(a,b)}\Big)\right|^q \mathrm{d}t \\
&\leq \int_h^1 \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\mathrm{d}t \\
&= (1-h) \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}. \qquad (16)
\end{aligned}
$$

By substituting (13)-(16) in (12), we get the desired result (11).

Theorem 2.3. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, for $h \in (0,1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ for any $n \geq 2$ the following inequality*

$$
\begin{aligned}
&\left|S_a^b(f)(h,n)\right| \\
&\leq ab(b-a)\left\{\Big(\frac{1+(nh-1)^{1+p}}{(1+p)n^{1+p}}\Big)^{\frac{1}{p}} \mu_{31}^{\frac{1}{q}}(h) \right. \\
&\qquad \left. + \Big(\frac{1+(n-nh-1)^{1+p}}{(1+p)n^{1+p}}\Big)^{\frac{1}{p}} \mu_{32}^{\frac{1}{q}}(h)\right\} \\
&\qquad \times \left\{\sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\right\}^{\frac{1}{q}} \qquad (17)
\end{aligned}
$$

holds, where

$$
\begin{aligned}
\mu_{31}(h) &= \frac{A_h^{1-2q}(a,b) - a^{1-2q}}{(1-2q)(b-a)}, \\
\mu_{32}(h) &= \frac{b^{1-2q} - A_h^{1-2q}(a,b)}{(1-2q)(b-a)}.
\end{aligned}
$$

Proof From Lemma 1 and by the Hölder integral inequality, we have

$$\begin{aligned}
&\left|S_a^b(f)(h,n)\right| \\
&\le ab(b-a)\Big[\int_0^h \frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t \\
&\quad+\int_h^1 \frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t\Big] \\
&\le ab(b-a)\Big[\Big\{\int_0^h \Big|t-\frac{1}{n}\Big|^p\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\int_0^h \frac{1}{A_t^{2q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\int_h^1 \Big|t-\frac{n-1}{n}\Big|^p\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\int_h^1 \frac{1}{A_t^{2q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].
\end{aligned} \tag{18}$$

Note that

$$\int_0^h \Big|t-\frac{1}{n}\Big|^p\mathrm{d}t=\frac{1+(nh-1)^{1+p}}{(1+p)n^{1+p}}, \tag{19}$$

$$\int_h^1 \Big|t-\frac{n-1}{n}\Big|^p\mathrm{d}t=\frac{1+(n-nh-1)^{1+p}}{(1+p)n^{1+p}}. \tag{20}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q>1$, we have

$$\begin{aligned}
&(i)\int_0^h \frac{1}{A_t^{2q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t \\
&\quad\le\Big(\int_0^h \frac{1}{A_t^{2q}(a,b)}\mathrm{d}t\Big)\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\} \\
&\quad=\mu_{31}(h)\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\},
\end{aligned} \tag{21}$$

$$\begin{aligned}
&(ii)\int_h^1 \frac{1}{A_t^{2q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t \\
&\quad\le\Big(\int_h^1 \frac{1}{A_t^{2q}(a,b)}\mathrm{d}t\Big)\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\} \\
&\quad=\mu_{32}(h)\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\}.
\end{aligned} \tag{22}$$

By substituting (19)-(22) in (18), we get the desired result (17).

Corollary 2.2. *In Theorem 2.3, one has:*

$$\left|S_a^b(f)(\frac{1}{2},6)\right| \leq ab(b-a)\Big\{\frac{1+2^{1+p}}{(1+p)6^{1+p}}\Big\}^{\frac{1}{p}}\Big\{\mu_{31}^{\frac{1}{q}}\big(\frac{1}{2}\big)+\mu_{32}^{\frac{1}{q}}\big(\frac{1}{2}\big)\Big\} \times\Big\{\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\}\Big\}^{\frac{1}{q}}. \tag{23}$$

Theorem 2.4. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for $h \in (0,1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$ for any $n \geq 2$ the following inequality*

$$\begin{aligned}&\left|S_a^b(f)(h,n)\right|\\ &\leq ab(b-a)\Big(\mu_{11}(h,n)+\mu_{12}(h,n)\Big)\\ &\qquad\times\Big(\sup\big\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\big\}\Big)^{\frac{1}{q}}.\end{aligned} \tag{24}$$

holds, where $\mu_{1i}(\cdot,\cdot)(i=1,2)$ are defined in Theorem 2.1.

Proof From Lemma 1 and by the power mean integral inequality, we have

$$\begin{aligned}&\left|S_a^b(f)(h,n)\right|\\ &\leq ab(a-b)\Big[\int_0^h\frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t\\ &\quad+\int_h^1\frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big|\Big)\mathrm{d}t\Big]\\ &\leq ab(a-b)\Big[\Big\{\int_0^h\frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\mathrm{d}t\Big\}^{\frac{1}{p}}\\ &\qquad\times\Big\{\int_0^h\frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\\ &\quad+\Big\{\int_h^1\frac{\left|t-\frac{n-1}{n}\right|}{A_t(a,b)}\mathrm{d}t\Big\}^{\frac{1}{p}}\\ &\qquad\times\Big\{\int_h^1\frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].\end{aligned} \tag{25}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$, we have

$$\begin{aligned}
(i) \int_0^h & \frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\left|f'\Big(\frac{ab}{A_t(a,b)}\Big)\right|^q \mathrm{d}t \\
&\leq \Big(\int_0^h \frac{\left|t-\frac{1}{n}\right|}{A_t^2(a,b)}\mathrm{d}t\Big) \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\} \\
&= \mu_{11}(h,n) \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}, \qquad (26) \\
(ii) \int_h^1 & \frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\left|f'\Big(\frac{ab}{A_t(a,b)}\Big)\right|^q \mathrm{d}t \\
&\leq \Big(\int_h^1 \frac{\left|t-\frac{n-1}{n}\right|}{A_t^2(a,b)}\mathrm{d}t\Big) \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\} \\
&= \mu_{12}(h,n) \sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}. \qquad (27)
\end{aligned}$$

By substituting (9)-(10) and (26)-(27) in (25), we get the desired result (24).

Corollary 2.3. *In Theorem 2.3, one has:*

$$\begin{aligned}
&\left|S_a^b(f)(\frac{1}{2},6)\right| \\
&\leq ab(b-a)\Big(\mu_{11}(\frac{1}{2},6)+\mu_{12}(\frac{1}{2},6)\Big) \\
&\qquad \times \Big(\sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\Big)^{\frac{1}{q}}. \qquad (28)
\end{aligned}$$

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